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ON ASYMPTOTIC LINEARITY OF L-ESTIMATES

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Dedicated to Professor Andrej Pázman on the occasion of his 70th birthday

(*Communicated by Gejza Wimmer*)

ABSTRACT. A theorem on asymptotic linearity of L-estimates is proved under general set of regularity conditions, allowing the sampled distribution to be non-integrable. The main result is the improvement in the order of the remainder term in the formula for asymptotic linearity of L-statistic. It is shown that in the case of the integral coefficients this term $R_n = \mathscr{O}_P(\frac{1}{n})$ and the case of functional coefficients is also covered.

1. Assumptions and main results

Suppose that $X_1, \dots, X_n$ is a random sample from the distribution of the random variable X and $X_n^{(1)} \leq \dots \leq X_n^{(n)}$ are the order statistics. The aim of the paper is to prove the asymptotic linearity of the L-estimate $\frac{1}{n} \sum\limits_{i=1}^{n} J(\frac{i}{n+1}) X_n^{(i)}$ in a way useful mainly in the case when X is not integrable. Before stating a theorem on this topic and discussing its relation with another results we present the regularity conditions imposed on distribution of X and on the score function J.

(A1) The distribution function $F(t) = P(X \leq t)$ is continuous and strictly increasing on (d, D), where $d = \inf\{t : F(t) > 0\}$, $D = \sup\{t : F(t) < 1\}$.

(A2) The function $J : (0,1) \to E^1$ possesses the derivative J' on $(0,1)$ and

$$\mu = \int_0^1 J(u) F^{-1}(u) \, \mathrm{d}u = E\big(J(F(X))x\big) \tag{1}$$

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is a real number.

(A3) There exist real numbers $\gamma = \gamma_d > -2$, $K > 0$ such that for each $u \in \left(0, \frac{1}{2}\right\rangle$

$$|J(u)| \le K u^{1+\gamma}\,, \quad |J'(u)| \le K u^{\gamma}\,.$$

There exist real numbers $\gamma = \gamma_D > -2$, $K > 0$ such that for each $u \in \left\langle \frac{1}{2}, 1\right)$

$$|J(u)| \le K(1-u)^{1+\gamma}\,, \quad |J'(u)| \le K(1-u)^{\gamma}\,.$$

(A4) There exist real numbers $\kappa_d < \gamma_d + 1$, $\kappa_D < \gamma_D + 1$ such that the integrals

$$\int_0^{1/2} u^{\kappa_d}\,\mathrm{d}F^{-1}(u) = \int_{-\infty}^{m} F^{\kappa_d}(x)\,\mathrm{d}x\,, \qquad \int_{1/2}^{1} (1-u)^{\kappa_D}\,\mathrm{d}F^{-1}(u) = \int_m^{\infty} (1-F(x))^{\kappa_D}\,\mathrm{d}x$$

are real numbers (here m denotes the median of F).

(A5) For every real number x the integrals

$$H(x) = \int_x^{+\infty} |J(F(y))|\,\mathrm{d}y\,, \qquad \int_{-\infty}^{+\infty} H(x)\,\mathrm{d}F(x), \qquad \int_{-\infty}^{+\infty} J(F(y))F(y)\,\mathrm{d}y$$

are real numbers and $F(x)H(x) \to 0$ as $|x| \to +\infty$.

Theorem 1. *Suppose that* (A1)–(A5) *hold and put*

$$\psi(x) = \int_{-\infty}^{+\infty} J(F(y))F(y)\,\mathrm{d}y - \int_x^{+\infty} J(F(y))\,\mathrm{d}y\,. \tag{2}$$

(I) *Let*

$$\tilde{L}_n = \sum_{i=1}^{n} \tilde{c}_{ni} X_n^{(i)}\,, \quad \tilde{c}_{ni} = \int_{\frac{i-1}{n}}^{\frac{i}{n}} J(u)\,\mathrm{d}u\,. \tag{3}$$

Then (cf. (1))

$$\tilde{L}_n = \mu + \frac{1}{n}\sum_{i=1}^{n} \psi(X_i) + \mathscr{O}_P\left(\frac{1}{n}\right)\,. \tag{4}$$

(II) *Put*

$$L_n^* = \sum_{i=1}^{n} c_{ni} X_n^{(i)}\,, \quad c_{ni} = \frac{1}{n} J\left(\frac{i}{n+1}\right)\,. \tag{5}$$

In addition to (A1)–(A5) *suppose also that for some* $t_d\,, t_D \in (d, D)$ *and some positive real numbers* β_d, β_D *the inequalities*

$$\begin{gathered} \sup\bigl\{|x|^{\beta_d} F(x):\ d < x \le t_d\bigr\} < +\infty\,, \\ \sup\bigl\{|x|^{\beta_D}(1-F(x)):\ t_D \le x < D\bigr\} < +\infty \end{gathered} \tag{6}$$

hold. If

$$\delta_1 = 2 + \gamma_d - \frac{1}{\beta_d} > 0\,, \quad \delta_2 = 2 + \gamma_D - \frac{1}{\beta_D} > 0\,, \tag{7}$$

then

$$L_n^* = \mu + \frac{1}{n}\sum_{i=1}^{n} \psi(X_i) + R_n\,, \tag{8}$$

where

$$R_n = \begin{cases} \mathscr{O}_P\left(\frac{\log n}{n}\right), & \min\{\delta_1, \delta_2\} = 1\,, \\ \mathscr{O}_P\left(\frac{1}{n^{\delta^*}}\right) & \text{otherwise}\,. \end{cases} \tag{9}$$

Here (cf. (7))

$$\delta^* = \min\{1\,,\, \delta_1\,, \delta_2\}\,. \tag{10}$$

Note that if $d < D$ from (A1) are real numbers, then (6), (7) hold, $F(x)H(x) \to 0$ as $|x| \to +\infty$ and (A3) implies that the integral (1) is finite.

Let the assumptions of the previous theorem be fulfiled. Then for the function (2) the equality $\int\limits_{-\infty}^{+\infty} \psi(x)\,\mathrm{d}F(x) = 0$ holds and if $V = \int\limits_{-\infty}^{+\infty} \psi^2(x)\,\mathrm{d}F(x) < +\infty$, by means of the central limit theorem one obtains that $\sqrt{n}(\tilde{L}_n - \mu) \to N(0, V)$ in distribution as n tends to infinity. Moreover, if also for the remainder term R_n from (9) the equality $R_n = o_P(n^{-1/2})$ holds, then $\sqrt{n}(L_n^* - \mu) \to N(0, V)$ in distribution.

A review of results on the asymptotic normality of L-estimates can be found in the monograph of Serfling [7]. General results on this topic are proved by Chernoff, Gastwirth and Johns [1] under set of conditions, which are of general nature but may be not easy to verify. The asymptotic linearity of L-estimates with the remainder term $R_n = \mathscr{O}_P(\frac{1}{n})$, from which the asymptotic normality follows by CLT, has been proved in Section 4 of Jurečková and Sen [3]. But for the L-statistics with the integral coefficients (3) and the number $\beta < 1$ from (6) they assume in their Theorem 4.3.1 that the untrimmed score function J fulfils the Lipschitz condition of the order $\nu > 1$, which in typical cases is not fulfilled, and for the statistics with the functional coefficients (5) the exponent β is in their Theorem 4.3.2 assumed to be greater than 1. Govindarajulu and Mason [2] proved the asymptotic linearity of the L-statistics even in a setting allowing X not to be integrable, but in difference from the previous theorem only with the remainder term $R_n = o_P(\frac{1}{\sqrt{n}})$. The remainder term in this paper has better accuracy than this result both for the L-statistics with integral coefficients and with functional scores as well, and the conditions (A1)-(A5) can be applied also in cases not covered by the conditions from Govindarajulu and Mason. Another results for strong representation of L-statistics were proved by Mason and Shorack [4] and [5], but again with $R_n = o_P\big(\frac{1}{\sqrt{n}}\big)$. Thus the main contribution of the previous theorem is the improvement in remainder term in the formula for the asymptotic linearity of the L-estimates in the case when $\beta < 1$, which occurs in the case when X is not integrable.

2. Proofs

In accordance with the assumptions of Theorem 1 throughout this section we assume that the assumptions (A1)–(A5) hold and we use the notation $F^{-1}(u) = \inf\{x : \ F(x) \geq u\}$. Further, $U_1, \dots, U_n$ denotes a random sample from the uniform distribution on $(0\,,\,1)$ and

$$U_n(t) = \frac{1}{n}\sum_{i=1}^{n} \chi_{\langle 0\,,\,t\rangle}(U_i)$$

its empirical distribution function. In the proofs we shall use the function

$$\phi(s) = -\int_s^1 J(u)\,\mathrm{d}u + \int_0^1 uJ(u)\,\mathrm{d}u\,, \quad s \in \langle 0\,,\,1\rangle\,, \tag{11}$$

and the fact, that under the validity of (A2) the equality $\phi^{'}(s) = J(s)$ holds. The symbol K will denote the generic constant, i.e., it will not depend on n but even though the symbol remains the same, it may denote various values.

P r o o f o f T h e o r e m 1(I). Since the condition (A4) holds, the integrals

$$\int_{-\infty}^{t} F(x)^{\gamma_d+2}\,\mathrm{d}x\,, \qquad \int_{t}^{+\infty} (1 - F(x))^{\gamma_D+2}\,\mathrm{d}x$$

are finite for every real number $t \in (d, D)$. Thus for the empirical distribution function

$$F_n(x) = \frac{1}{n}\sum_{j=1}^{n} \chi_{(-\infty,x\rangle}(X_j)$$

the integral $\int\limits_{-\infty}^{+\infty} (\phi(F_n(x)) - \phi(F(x)))\,\mathrm{d}x$ is finite, and making use of the integration by parts and proceeding similarly as described in [3, p. 144], one obtains that

$$\tilde{L}_n - \mu - \frac{1}{n}\sum_{i=1}^{n}\psi(X_i) = \int_0^1 V_n(s)\,\mathrm{d}F^{-1}(s)\,,$$

where

$$|V_n(s)| = |\phi(U_n(s)) - \phi(s) - \phi'(s)(U_n(s) - s)|\,.$$

By [6, Theorem 2.11.10] given $\eta \in (0, \frac{1}{2})$ there exists a constant $C(\eta)$ such that for every real number $M > 0$

$$P\biggl(\sup_{0<s<1} \frac{\Bigl|\sqrt{n}\Bigl(U_n(s) - s\Bigr)\Bigr|}{(s(1-s))^{1/2-\eta}} > M\biggr) \le \frac{C(\eta)}{M^2}$$

for all integers $n > 1$. Hence if $\varepsilon > 0$ and $\eta \in (0\,, 1/2)$ then there exists a number $M > 0$ such that with probability at least $1 - \varepsilon$

$$\sup_{0<s<1} \frac{\Bigl|\sqrt{n}\Bigl(U_n(s) - s\Bigr)\Bigr|}{(s(1-s))^{1/2-\eta}} \le M\,. \tag{12}$$

Further, employing the Daniels theorem from [8, p. 345], one obtains that

$$\sup_{s\in(0,1)} \frac{U_n(s)}{s} = \mathscr{O}_P(1)\,, \qquad \sup_{s\in(0,1)} \frac{1-U_n(s)}{1-s} = \mathscr{O}_P(1)\,,$$

the Wellner-Shorack inequality from [8, p. 415] implies that

$$\sup\left\{\frac{s}{U_n(s)}:\ U_n(s) > 0\,,\ s < 1\right\} = \mathscr{O}_P(1)\,,$$
$$\sup\left\{\frac{1-s}{1-U_n(s)}:\ U_n(s) < 1\,,\ s > 0\right\} = \mathscr{O}_P(1)\,.$$

Hence given $\varepsilon > 0$ there exist positive constants a_1, a_2, b_1, b_2 such that for all n with probability at least $1 - \varepsilon$

$$U_n(s) > 0\,,\ s \in (0,1) \implies a_1 s < U_n(s) < a_2 s\,, \tag{13}$$
$$U_n(s) < 1\,,\ s \in (0,1) \implies b_1(1-s) < 1 - U_n(s) < b_2(1-s)\,. \tag{14}$$

Thus it is sufficient to prove that for a suitably chosen $\eta \in (0\,, \frac{1}{2})$ under the validity of (12), (13) and (14)

$$\int_0^1 |V_n(s)|\,\mathrm{d}F^{-1}(s) = \mathscr{O}\Bigl(\frac{1}{n}\Bigr)\,. \tag{15}$$

In proving this we shall utilize the fact, that for any $0 \le \alpha \le 1$ and positive real numbers c_1, c_2 the inequality

$$(\alpha c_1 + (1-\alpha)c_2)^\gamma \le c_1^\gamma + c_2^\gamma \tag{16}$$

holds.

Let $0 < \delta < 1/2$ be a fixed real number, $s \in (0, \delta)$ and $\gamma = \gamma_d$. Assume that $U_n(s) > 0$. An application of Taylor theorem, (A3), (12), (13) and (16) yields that

$$\begin{aligned} |V_n(s)| &= \Big| J'\Big(\alpha s + (1-\alpha)U_n(s)\Big)\Big| \frac{(U_n(s)-s)^2}{2} \\ &\le \frac{K}{n}\Big(\alpha s + (1-\alpha)U_n(s)\Big)^{\gamma} s^{2(1/2-\eta)} \\ &\le \frac{K}{n}\Big(s^{\gamma} + U_n(s)^{\gamma}\Big) s^{1-2\eta} \\ &\le \frac{K}{n} s^{\gamma+1-2\eta} \,. \end{aligned}$$

Similarly if $U_n(s) = 0$, then

$$\begin{aligned} |V_n(s)| &= |\phi(0) - \phi(s) + \phi'(s)s| \\ &\le \int_0^s |J(u)| \,\mathrm{d}u + K s^{\gamma+2} \\ &\le K \int_0^s u^{\gamma+1} \,\mathrm{d}u + K s^{\gamma+2} \le K s^{\gamma+2} \\ &= K s^{\gamma}(s - U_n(s))^2 \le \frac{K}{n} s^{\gamma+1-2\eta} \,. \end{aligned}$$

Thus owing to (A4)

$$\int_0^{\delta} |V_n(s)| \,\mathrm{d}F^{-1}(s) \le \int_0^{\delta} \frac{K}{n} s^{\gamma+1-2\eta} \,\mathrm{d}F^{-1}(s) \le \frac{K}{n} \,.$$

Since for $s \in (\delta, 1)$ one can proceed similarly, (15) is proved. □

In the rest of the section we assume that in addition to (A1)–(A5) also the inequalities (6), (7) hold for some positive real β_d, β_D and some $t_d\,, t_D \in (d, D)$.

The proof of the assertion (II) of the theorem from the previous section will be based on the following auxiliary assertions.

Lemma 1. *Let $U_n^{(j)}$ denotes the jth order statistic from $U_1, \dots, U_n$. Then for every positive real number c*

$$\lim_{n\to\infty} P\left(U_n^{(1)} < \frac{c}{n}\right) = 1 - \mathrm{e}^{-c}\,, \qquad \lim_{n\to\infty} P\left(U_n^{(n)} > 1 - \frac{c}{n}\right) = 1 - \mathrm{e}^{-c}\,.$$

Lemma 2. *For each $u \in (0,1)$*

$$|F^{-1}(u)| \le \max\left\{ \frac{K}{u^{1/\beta_d}}\,, \frac{K}{(1-u)^{1/\beta_D}} \right\} .$$

Lemma 3. *Suppose that the number* (cf. (7))

$$\delta = \min\{\delta_1\,,\delta_2\} \tag{17}$$

is positive. For $c>0$ put

$$h_n(c) = \frac{c}{n}\,, \qquad H_n(c) = 1 - \frac{c}{n}\,.$$

(I) *The equality*

$$\int\limits_0^{h_n(c)} |J(u)F^{-1}(u)|\,\mathrm{d}u + \int\limits_{H_n(c)}^{1} |J(u)F^{-1}(u)|\,\mathrm{d}u = \mathscr{O}\Big(\frac{1}{n^{\delta}}\Big) \tag{18}$$

holds.

(II) *Define the function $J_n(u)$ on $(0,1)$ by the formula*

$$J_n(u) = J\Big(\frac{i}{n+1}\Big) \qquad \textit{if} \quad \frac{i-1}{n} < u \le \frac{i}{n}\,, \quad i = 1,\dots,n\,, \tag{19}$$

and put

$$R_n^{(1)} = \int\limits_{h_n(c)}^{1/2} |J_n(u) - J(u)||F^{-1}(u)|\,\mathrm{d}u\,, \quad R_n^{(2)} = \int\limits_{1/2}^{H_n(c)} |J_n(u) - J(u)||F^{-1}(u)|\,\mathrm{d}u\,. \tag{20}$$

Then with the notation from (7)

$$R_n^{(1)} = \begin{cases} \mathscr{O}\Big(\frac{1}{n^{\delta_1^*}}\Big)\,,\ \delta_1^* = \min\{1\,,\,\delta_1\} & \delta_1 \ne 1\,, \\ \mathscr{O}\Big(\frac{\log n}{n}\Big) & \delta_1 = 1\,, \end{cases} \tag{21}$$

$$R_n^{(2)} = \begin{cases} \mathscr{O}\Big(\frac{1}{n^{\delta_2^*}}\Big)\,,\ \delta_2^* = \min\{1\,,\,\delta_2\} & \delta_2 \ne 1\,, \\ \mathscr{O}\Big(\frac{\log n}{n}\Big) & \delta_2 = 1\,. \end{cases} \tag{22}$$

P r o o f. The proof of (I) easily follows from (A3), (6) and (7). If $\lambda \in (0,1)$, then one can prove by means of (A3) and (16) that for each $u \in (\frac{c}{n}\,,\,\lambda\rangle$ the inequalities

$$|J_n(u) - J(u)| \le \frac{K}{n}\left(\left(\frac{\lfloor nu\rfloor + 1}{n+1}\right)^{\gamma_d} + u^{\gamma_d}\right) \le \frac{K}{n}u^{\gamma_d}\,, \quad K = K(c,\lambda) \tag{23}$$

hold (here $\lfloor a\rfloor$ denotes the largest integer not exceeding a). Employing (6) and (7) after some computation one obtains the formula (21), (22) can be proved analogously. □

Lemma 4. *Suppose that I_s denotes for $s \in (0,1)$ the interval with the endpoints s, $U_n(s)$, i.e., $I_s = (s, U_n(s))$ if $s < U_n(s)$ and $I_s = (U_n(s), s)$ otherwise. Then in the notation from the previous lemmas*

$$\int_{U_n^{(1)}}^{U_n^{(n)}} \left(\int_{I_s} |J_n(u) - J(u)| \, \mathrm{d}u \right) \mathrm{d}F^{-1}(s) = \mathcal{O}_P\Big(\frac{1}{n}\Big) .$$

P r o o f. Lemma 1 implies that given $\varepsilon > 0$ there exists a positive constant c such that for all sample sizes n sufficiently large

$$\frac{1}{2} > U_n^{(1)} \geq \frac{c}{n}, \quad \frac{1}{2} < U_n^{(n)} \leq 1 - \frac{c}{n} \tag{24}$$

with probability at least $1 - \varepsilon$. Therefore we may assume that the inequalities (24), (13) and (14) are fulfilled. Further, according to the Glivenko-Cantelli theorem we may assume that for all $n \geq n_0$ and $s \in \langle 0, \frac{1}{2} \rangle$ the inequalities $U_n(s) \leq \frac{1}{2} + \varepsilon^* < 1$ hold. Thus employing (24) and (13) we obtain the validity of (23) on the interval I_s for each $s \in (U_n^{(1)}, \frac{1}{2})$, and the repeated use of (13) yields that

$$\int_{U_n^{(1)}}^{1/2} \left(\int_{I_s} |J_n(u) - J(u)| \, \mathrm{d}u \right) \mathrm{d}F^{-1}(s) \leq \int_{U_n^{(1)}}^{1/2} \frac{K}{n} s^{\gamma_d + 1} \, \mathrm{d}F^{-1}(s) = \mathcal{O}_P\Big(\frac{1}{n}\Big) ,$$

where the last equality follows from (A4). Since the statement

$$\int_{1/2}^{U_n^{(n)}} \left(\int_{I_s} |J_n(u) - J(u)| \, \mathrm{d}u \right) \mathrm{d}F^{-1}(s) = \mathcal{O}_P\Big(\frac{1}{n}\Big)$$

can be verified similarly, the lemma is proved. □

P r o o f o f T h e o r e m 1(II). Let F_n denote the empirical distribution function of $X_1, \dots, X_n$ and (cf. (19))

$$\phi_n(s) = -\int_s^1 J_n(u) \, \mathrm{d}u + \int_0^1 u J_n(u) \, \mathrm{d}u , \quad s \in \langle 0, 1 \rangle .$$

Then for the statistic (5) the equality

$$L_n^* = \int_{-\infty}^{+\infty} x \, \mathrm{d}\phi_n(F_n(x)) \tag{25}$$

holds. Put $U_i = F(X_i)$, $i = 1, \dots, n$. As the set having the probability not exceeding ε can be neglected, according to Lemma 1 we may assume that for properly chosen positive $c_1 < c_2$

$$\begin{gathered} F^{-1}(\tfrac{c_1}{n}) < X_n^{(1)} < F^{-1}(\tfrac{c_2}{n}), \quad F^{-1}(1-\tfrac{c_2}{n}) < X_n^{(n)} < F^{-1}(1-\tfrac{c_1}{n}), \\ X_i = F^{-1}(F(X_i)), \quad i = 1, \dots, n. \end{gathered} \tag{26}$$

Put

$$\mu_n = \int_{U_n^{(1)}}^{U_n^{(n)}} J_n(u) F^{-1}(u)\, \mathrm{d}u\,.$$

By means of the continuity of F

$$\mu_n = \int_{F^{-1}(U_n^{(1)})}^{F^{-1}(U_n^{(n)})} J_n(F(x)) x\, \mathrm{d}F(x) = \int_{F^{-1}(U_n^{(1)})}^{F^{-1}(U_n^{(n)})} x\, \mathrm{d}\phi_n(F(x)),$$

because for $a < b$

$$\phi_n(F(b)) - \phi_n(F(a)) = \int_{F(a)}^{F(b)} J_n(u)\, \mathrm{d}u = \int_a^b J_n(F(x))\, \mathrm{d}F(x)\,.$$

Since the product of right-continuous functions of bounded variation has also this property, the function $G(x) = x[\phi_n(F_n(x)) - \phi_n(F(x))]$ induces a signed measure ν_G. Thus making use of the integration by parts one obtains

$$L_n^* - \mu_n = \int_{\langle X_n^{(1)}, X_n^{(n)}\rangle} x\, \mathrm{d}\Big[\phi_n(F_n(x)) - \phi_n(F(x))\Big] = \nu_G(\langle X_n^{(1)}, X_n^{(n)}\rangle) - T_2\,,$$

where the second term

$$\begin{aligned} T_2 &= \int_{X_n^{(1)}}^{X_n^{(n)}} \Big[\phi_n(F_n(x)) - \phi_n(F(x))\Big]\, \mathrm{d}x \\ &= \int_{F(X_n^{(1)})}^{F(X_n^{(n)})} \Big[\phi_n(F_n(F^{-1}(s))) - \phi_n(F(F^{-1}(s)))\Big]\, \mathrm{d}F^{-1}(s) \\ &= \int_{U_n^{(1)}}^{U_n^{(n)}} \Big[\phi_n(U_n(s)) - \phi_n(s)\Big]\, \mathrm{d}F^{-1}(s). \end{aligned}$$

Hence if we show that

$$\nu_G(\langle X_n^{(1)}, X_n^{(n)}\rangle) = \mathscr{O}_P\Big(\frac{1}{n^\delta}\Big), \tag{27}$$

where δ is defined in (17), we obtain that

$$L_n^* - \mu_n = \mathscr{O}_P\Big(\frac{1}{n^\delta}\Big) - \int\limits_{U_n^{(1)}}^{U_n^{(n)}} \Big(\phi_n(U_n(s)) - \phi_n(s)\Big)\, \mathrm{d}F^{-1}(s). \tag{28}$$

But

$$G(X_n^{(n)}) = X_n^{(n)}[\phi_n(1) - \phi_n(F(X_n^{(n)}))] = F^{-1}(U_n^{(n)}) \int\limits_{U_n^{(n)}}^{1} J_n(u)\, \mathrm{d}u, \tag{29}$$

$$\begin{aligned} G(X_n^{(1)})^- &= \lim_{z \nearrow X_n^{(1)}} z[\phi_n(0) - \phi_n(F(z))] = X_n^{(1)}[\phi_n(0) - \phi_n(F(X_n^{(1)}))] \\ &= -F^{-1}(U_n^{(1)}) \int\limits_0^{U_n^{(1)}} J_n(u)\, \mathrm{d}u. \end{aligned} \tag{30}$$

Assume without the loss of generality that $c_2 > 2$. Then by (A3)

$$\begin{aligned} \int\limits_{U_n^{(n)}}^{1} |J_n(u)|\, \mathrm{d}u \le \int\limits_{1-\frac{c_2}{n}}^{1} |J_n(u)|\, \mathrm{d}u \le \sum_{i=n-[c_2]}^{n} \int\limits_{\frac{i-1}{n}}^{\frac{i}{n}} |J_n(u)|\, \mathrm{d}u \\ \le \frac{K}{n} \sum_{i=n-[c_2]}^{n} \Big(1 - \frac{i}{n+1}\Big)^{1+\gamma_D} \\ \le \frac{K}{n} \sum_{i=1}^{[c_2]+1} \frac{i^{1+\gamma_D}}{n^{1+\gamma_D}} \le \frac{K}{n^{2+\gamma_D}}. \end{aligned}$$

This together with (29), (26) and Lemma 2 means, that

$$|G(X_n^{(n)})| \le |F^{-1}(U_n^{(n)})| \int\limits_{U_n^{(n)}}^{1} |J_n(u)|\, \mathrm{d}u \le \frac{K}{(1 - U_n^{(n)})^{\frac{1}{\beta_D}}} \frac{1}{n^{2+\gamma_D}} \le \frac{K}{n^{2+\gamma_d - 1/\beta_D}}. \tag{31}$$

Further, since according to (26) the inequality $U_n^{(1)} \le \frac{c_2}{n}$ holds, by means of (A2)

$$\int\limits_0^{U_n^{(1)}} |J_n(u)|\, \mathrm{d}u \le \int\limits_0^{\frac{[c_2]+1}{n}} |J_n(u)|\, \mathrm{d}u \le \frac{K}{n} \sum_{i=1}^{[c_2]+1} \Big(\frac{i}{n+1}\Big)^{1+\gamma_d} \le \frac{K}{n^{2+\gamma_d}}$$

Combining this with (30), (26) and Lemma 2 one obtains

$$|G(X_n^{(1)})^-| \leq |F^{-1}(U_n^{(1)})| \int_0^{U_n^{(1)}} |J_n(u)| \,\mathrm{d}u \leq \frac{K}{(U_n^{(1)})^{\frac{1}{\beta_d}}} \frac{1}{n^{2+\gamma_d}} \leq \frac{K}{n^{2+\gamma_d-\frac{1}{\beta_d}}}. \tag{32}$$

Obviously, (31) and (32) imply (27) Further, $\tilde{L}_n = \int x \,\mathrm{d}\phi(F_n(x))$, where ϕ is defined in (11). This together with (18), (26) and integration by parts similarly as in (28) means that

$$\tilde{L}_n - \mu = \mathscr{O}_P\Big(\frac{1}{n^\delta}\Big) - \int_{U_n^{(1)}}^{U_n^{(n)}} \Big[\phi(U_n(s)) - \phi(s)\Big] \,\mathrm{d}F^{-1}(s)\,. \tag{33}$$

Taking into account (28), (33) and Lemma 4 one obtains that

$$\begin{aligned} |(L_n^* - \mu_n) - (\tilde{L}_n - \mu)| &\leq \mathscr{O}_P\Big(\frac{1}{n^\delta}\Big) + \int_{U_n^{(1)}}^{U_n^{(n)}} \Big[\int_{I_s} |J_n(u) - J(u)| \,\mathrm{d}u\Big] \,\mathrm{d}F^{-1}(s) \\ &= \mathscr{O}_P\Big(\frac{1}{n^{\delta^*}}\Big), \end{aligned} \tag{34}$$

where δ^* is defined in (10). But by means of (26) and Lemma 3

$$|\mu_n - \mu| \leq \int_{U_n^{(1)}}^{U_n^{(n)}} |J_n(u) - J(u)|\, |F^{-1}(u)| \,\mathrm{d}u + \mathscr{O}\Big(\frac{1}{n^\delta}\Big) \leq R_n^{(1)} + R_n^{(2)} + \mathscr{O}\Big(\frac{1}{n^\delta}\Big), \tag{35}$$

where $R_n^{(1)}$, $R_n^{(2)}$ are defined by (20) with $c = c_1$, and (8)–(10) can be obtained from (34), (35), (21), (22) and (4). □

REFERENCES

[1] CHERNOFF, H.—GASTWIRTH, J. L.—JOHNS, M. V. JR.: *Asymptotic distribution of linear combinations of order statistics, with applications to estimation*, Ann. Math. Statist. **38** (1967), 52–72.

[2] GOVINDARAJULU, Z.—MASON, D. M.: *A strong representation for linear combinations of order statistics with application to fixed-width confidence intervals for location and scale parameters*, Scand. J. Statist. **10** (1983), 97–115.

[3] JUREČKOVÁ, J.—SEN, P. K: *Robust Statistical Procedures. Asymptotics and Interrelations*, John Wiley & Sons, New York, 1996.

[4] MASON, D. M.—SHORACK, G. R.: *Neccessary and sufficient conditions for asymptotic normality of trimmed L statistics*, J. Statist. Plann. Inference **25** (1990), 111–139.

[5] MASON, D. M.—SHORACK, G. R.: *Neccessary and sufficient conditions for asymptotic normality of L statistics*, Ann. Probab. **20** (1992), 1779–1804.

[6] PURI, M. L.—SEN, P. K.: *Nonparametric Methods in Multivariate Analysis*, John Wiley & Sons, New York, 1971.
[7] SERFLING, R. J.: *Approximation Theorems of Mathematical Statistics*, John Wiley & Sons, New York, 1980.
[8] SHORACK, G. W.—WELLNER, J. A.: *Empirical Processes with Applications to Statistics*, John Wiley & Sons, New York, 1986.

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