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ASYMPTOTIC LOCAL POWER OF THE LR TEST FOR SOME HOMOGENEITY HYPOTHESES ON NORMAL DISTRIBUTIONS

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ABSTRACT. Explicit formulas for the non-centrality parameters of the asymptotic chi-square distribution of the LR statistic are presented for testing the hypothesis of equality of the means and of the covariance matrices, and also for the Behrens-Fisher problem of testing the equality of means without any restriction on covariances of the underlying normal distributions.

Suppose that the probabilities $\{\,\overline{P}_\gamma;\, \gamma \in \Xi\,\}$ are defined by means of densities $\{f(x,\gamma);\, \gamma \in \Xi\}$ with respect to a measure ν on $(X, \mathcal{S})$, and $x(j, n_j)$ denotes random sample of size n_j from jth distribution $\overline{P}_{\theta_j}$, $j = 1, \dots, q$. Thus the pooled random sample $x_{(n_1,\dots,n_q)} = (x(1,n_1), x(2,n_2), \dots, x(q,n_q))$, the parametric set describing its distribution $\Theta = \Xi^q$ and in $\theta = (\theta_1, \dots, \theta_q) \in \Theta$ the jth component θ_j denotes the parameter of the jth population.

Consider testing of the hypothesis $\Omega \subset \Theta$ by $T_{n_1,\dots,n_q} = T_{n_1,\dots,n_q}(x_{(n_1,\dots,n_q)})$ with large values significant (the hypothesis is rejected if $T > t$). We shall deal with the likelihood ratio test statistic

$$T_{n_1,\dots,n_q} = 2 \log \frac{L\big(x_{(n_1,\dots,n_q)}, \Theta\big)}{L\big(x_{(n_1,\dots,n_q)}, \Omega\big)}\,, \tag{1}$$

where with the notation $x(j, n_j) = \left(x_1^{(j)}, \dots, x_{n_j}^{(j)}\right)$,

$$L\big(x_{(n_1,\dots,n_q)}, \Omega\big) = \sup\left\{ \prod_{j=1}^{q} \prod_{i=1}^{n_j} f\big(x_i^{(j)}, \theta_j\big);\, \theta = (\theta_1, \dots, \theta_q) \in \Omega \right\}. \tag{2}$$

Quality of the test can be judged by various criteria, one is the value of exact slope. LR statistic (1) is shown to be optimal in the sense of exact slopes

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for every null hypothesis closed in the parameter set Θ for various probability classes, including the normal distributions, in the papers [5], [6]. Another possible criterion is the value of the noncentrality parameter of the limiting chi-square distribution of test statistic when the true parameters are Pitman alternatives. Since LR test is optimal in the sense of exact slopes, its noncentrality parameter can be used as a base for measuring the asymptotic efficiency of various tests. General formulas for the noncentrality parameter of the LR statistic in this multisample setting are in [8] derived by means of the following conditions.

(C 1) Ξ is an open subset of R^m, for each $x \in X$ there exist partial derivatives $\frac{\partial^2 f(x,\gamma)}{\partial\gamma_i\partial\gamma_j}$, $i,j = 1,\dots,m$ and they are continuous on Ξ.

(C 2) The equality $\int \frac{\partial^2 f(x,\gamma)}{\partial\gamma_i\partial\gamma_j}\,\mathrm{d}\nu(x) = 0$ holds for all $\gamma \in \Xi$ and $i,j = 1,\dots,m$.

(C 3) The function $f(.,.)$ is positive on $X \times \Xi$ and for each parameter $\gamma \in \Xi$ there exist a $\overline{P}_\gamma$ integrable function h_γ and a neighbourhood $U_\gamma \subset \Xi$ of the point γ such that the inequality

$$\left| \frac{\partial^2 \log f(x,\gamma^*)}{\partial\gamma_i^*\partial\gamma_j^*} \right| \le h_\gamma(x)$$

holds for all $\gamma^* \in U_\gamma$, $x \in X$ and $i,j = 1,\dots m$.

(C 4) For every $\gamma \in \Xi$ the function $\frac{\partial \log f(x,\gamma)}{\partial\gamma} = \left(\frac{\partial \log f(x,\gamma)}{\partial\gamma_1},\dots,\frac{\partial \log f(x,\gamma)}{\partial\gamma_m}\right)^T$ belongs to $\mathcal{L}_2(\overline{P}_\gamma)$ and the matrix

$$\mathbf{J}(\gamma) = \left(E_\gamma\left(\frac{\partial \log f(x,\gamma)}{\partial\gamma_i}\,\frac{\partial \log f(x,\gamma)}{\partial\gamma_j} \right)\right)_{i,j=1}^m$$

is positive definite and continuous on Ξ.

(C 5) Let $\overline{P}_\gamma^{(n)}$ be the product measure of n copies of $\overline{P}_\gamma$ and $L(x_1,\dots,x_n,\gamma) = \prod_{i=1}^n f(x_i,\gamma)$. There exist measurable mappings $\hat\gamma_n : X^n \to \Xi$ such that for each parameter $\gamma \in \Xi$ and every real number $\varepsilon > 0$,

$$\lim_{n\to\infty} \overline{P}_\gamma^{(n)}\big\{\|\hat\gamma_n(x_1,\dots,x_n) - \gamma\| \ge \varepsilon\big\} = 0,$$

$$\lim_{n\to\infty} \overline{P}_\gamma^{(n)}\Big\{L(x_1,\dots,x_n,\Xi) = L\big(x_1,\dots,x_n,\hat\gamma_n(x_1,\dots,x_n)\big)\Big\} = 1.$$

Let $u = 1,2,\dots$ denote the index of the experiment, i.e., the sample size from the jth population $n_j = n_j^{(u)}$, $j = 1,\dots,k$, and total sample size

$$N_u = n_1^{(u)} + \dots + n_q^{(u)}. \tag{3}$$

Now suppose that the relative sample sizes

$$\hat{p}_j = \frac{n_j^{(u)}}{N_u} \to p_j > 0\,, \quad n_j^{(u)} \to +\infty\,, \qquad j = 1, \dots, q \tag{4}$$

as $u \to \infty$. Let the null hypothesis

$$\Omega = \left\{ \theta^* \in \Theta;\, g_1(\theta^*) = 0, \dots, g_{k_0}(\theta^*) = 0 \right\}, \tag{5}$$

and the functions $g_j : \Theta \to R^1$ belong to $\mathcal{C}_1$. Further, assume that the mq-dimensional parameter θ belongs to Ω and the matrix

$$\boldsymbol{\partial}_{\mathbf{0}}(\theta) = \begin{pmatrix} \dfrac{\partial g_1(\theta)}{\partial \theta_1}, & \dots & , \dfrac{\partial g_1(\theta)}{\partial \theta_{mq}} \\ \vdots & & \vdots \\ \dfrac{\partial g_{k_0}(\theta)}{\partial \theta_1}, & \dots & , \dfrac{\partial g_{k_0}(\theta)}{\partial \theta_{mq}} \end{pmatrix} \tag{6}$$

is of rank k_0. For $h = \left(h_1^T, \dots, h_q^T\right)^T \in R^{mq}$, where $h_j \in R^m$ for all j, let $\pi_j(h) = h_j$ denote the projection onto the jth coordinate space R^m. Suppose that the vectors $\{h_u\}_{u=1}^{\infty}$ from R^{mq} are such that

$$\lim_{u \to \infty} h_u = h \in R^{mq}. \tag{7}$$

If the product measure corresponding to the uth experiment (cf. (3))

$$P_u^* = \overline{P}_{\theta(1,u)}^{(n_1^{(u)})} \times \dots \times \overline{P}_{\theta(q,u)}^{(n_q^{(u)})}\,, \qquad \theta(j,u) = \pi_j(\theta) + \frac{\pi_j(h_u)}{\sqrt{N_u}}\,, \tag{8}$$

i.e., in the uth experiment the size of the sample from the jth population is $n_j^{(u)}$ and this sample is drawn from $\overline{P}_{\theta(j,u)}$ with $\theta(j,u)$ described in (8), then application of Corollary 1.2, p. 583 of [8] yields that for the LR statistic (cf. (1))

$$T_u = T_{n_1^{(u)}, \dots, n_q^{(u)}} \tag{9}$$

the weak convergence of probabilities

$$\mathcal{L}\Big[\, T_u \,\Big|\, P_u^* \Big] \longrightarrow \chi^2_{k_0}(\lambda) \tag{10}$$

to the chi-square distribution with k_0 degrees of freedom and the noncentrality parameter λ holds as $u \to \infty$, and (cf. (4), (7))

$$\begin{gathered} \lambda = h^T \boldsymbol{\partial}_{\mathbf{0}}(\theta)^T \Big(\mathbf{F}_0 \mathbf{J}(\theta)^{-1} \mathbf{F}_0^T\Big)^{-1} \boldsymbol{\partial}_{\mathbf{0}}(\theta) h\,, \qquad \mathbf{F}_0 = \boldsymbol{\partial}_{\mathbf{0}}(\theta) \mathbf{D}(\mathbf{p})^{-1/2}\,, \\ \mathbf{D}(\mathbf{p})^{1/2} = \mathrm{diag}\big(\sqrt{p_1} I_m, \dots, \sqrt{p_q} I_m\big)\,, \quad \mathbf{J}(\theta) = \mathrm{diag}\Big(\mathbf{J}\big(\pi_1(\theta)\big), \dots, \mathbf{J}\big(\pi_q(\theta)\big)\Big)\,, \end{gathered} \tag{11}$$

provided that the regularity conditions (C 1)–(C 5) are fulfiled and with probability tending to 1 there exists ML estimator of the unknown parameter $\theta \in \Omega$

(i.e., under the restrictions Ω) which is consistent under the validity of the null hypothesis Ω (I_m in (11) denotes the $m \times m$ identity matrix). We remark that the results in [8] are proved by means of the Le Cam lemmas on the contiguity of probabilities and the asymptotic distribution under the local alternatives, and by means of the results on the uniform LAN property from the monograph [1]. An important tool of proofs carried out in [8] is also the concept of the set sequentially approximable by the cone, presented in [2] and [3].

In what follows, the vector $e(\mathbf{\Sigma}) = (\Sigma_{11}, \dots, \Sigma_{kk}, \Sigma_{12}, \dots, \Sigma_{1k}, \dots, \Sigma_{k-1k})^T$ denotes the elements of symmetric $k \times k$ matrix $\mathbf{\Sigma}$ which are not below the diagonal, and

$$\Xi = \left\{ \begin{pmatrix} \boldsymbol{\mu} \\ e(\mathbf{\Sigma}) \end{pmatrix}; \quad \boldsymbol{\mu} \in R^k, \quad \mathbf{\Sigma} \textit{ is symmetric positive definite } k \times k \textit{ matrix} \right\}, \tag{12}$$

$$f(\mathbf{x}, \boldsymbol{\gamma}) = \frac{1}{(2\pi)^{k/2} |\mathbf{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \mathbf{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right), \quad \boldsymbol{\gamma} = \left(\boldsymbol{\mu}^T, e(\mathbf{\Sigma})^T \right)^T, \tag{13}$$

where $|\mathbf{\Sigma}|$ denotes the determinant of $\mathbf{\Sigma}$, denote the parameter set of k-dimensional normal distributions and their densities.

Lemma 1. *Let* $\{f(\mathbf{x}, \boldsymbol{\gamma}); \gamma \in \Xi\}$ *be the family of normal densities* (13) *indexed by the set* (12). *Then the regularity conditions* (C 1)–(C 5) *hold and for* $\gamma = \left(\boldsymbol{\mu}^T, e(\mathbf{\Sigma})\right)^T \in \Xi$ *the Fisher information matrix*

$$\mathbf{J}(\gamma) = \begin{pmatrix} \mathbf{\Sigma}^{-1} & \mathbf{0}_{k \times c} \\ \mathbf{0}_{c \times k} & \mathbf{V} \end{pmatrix}, \tag{14}$$

where $c = k(k+1)/2$ *and* $\mathbf{V}$ *is a regular* $c \times c$ *matrix.*

P r o o f. The conditions (C 1)–(C 3) can be verified by means of the formulas for derivatives of the determinant and of the inverse matrix, validity of (C 5) is well-known and the rest of the proof follows from Theorem 2 on pp. 317–318 of [4] and from Theorem 14 on p. 50 of [4]. □

Throughout the rest of the text suppose that k is a positive integer, θ_j denotes the parameter of the jth normal distribution $N_k(\boldsymbol{\mu}_j, \mathbf{\Sigma}_j)$, $j = 1, \dots, q$, i.e., $\Theta = \Xi^q$ is the set of overall parameters of the sampled normal populations, and for the relative sample sizes (4) holds.

Theorem 2. *Suppose that the null hypothesis*

$$\Omega = \left\{ (\theta_1, \dots, \theta_q);\ \boldsymbol{\mu}_1 = \dots = \boldsymbol{\mu}_q,\ \mathbf{\Sigma}_1 = \dots = \mathbf{\Sigma}_q \right\} \tag{15}$$

and that $\theta = (\theta_1, \ldots, \theta_q)$ belongs to Ω, i.e., $\theta_1 = \ldots = \theta_q$ is the parameter of $N_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ distribution. If the Pitman alternatives $\theta(j,u) = \theta_j + \pi_j(\delta_u)$, where

$$\delta_u = \frac{1}{\sqrt{N_u}}\left(\begin{pmatrix} \mathbf{M}_1 \\ e(\mathbf{W}_1) \end{pmatrix}, \ldots, \begin{pmatrix} \mathbf{M}_q \\ e(\mathbf{W}_q) \end{pmatrix}\right),$$

π_j denotes the projection onto the jth component, $\mathbf{M}_j \in R^k$ and $\mathbf{W}_j$ is a symmetric $k \times k$ matrix, $j = 1, \ldots, k$, then for the LR statistic (cf. (9), (8))

$$\mathcal{L}\Big[\, T_u \,\big|\, P_u^* \Big] \longrightarrow \chi^2_{(q-1)m}(\lambda) \tag{16}$$

as $u \to \infty$, where $m = k + \frac{k(k+1)}{2}$ and the noncentrality parameter

$$\lambda = \sum_{j=1}^{q} p_j \big(\mathbf{M}_j - \overline{\mathbf{M}}\big)^T \boldsymbol{\Sigma}^{-1} \big(\mathbf{M}_j - \overline{\mathbf{M}}\big) + \sum_{j=1}^{q} p_j e\big(\mathbf{W}_j - \overline{\mathbf{W}}\big)^T \mathbf{V} e\big(\mathbf{W}_j - \overline{\mathbf{W}}\big)\,, \tag{17}$$

$$\overline{\mathbf{M}} = \sum_{j=1}^{q} p_j \mathbf{M}_j\,, \quad \overline{\mathbf{W}} = \sum_{j=1}^{q} p_j \mathbf{W}_j \tag{18}$$

and $\mathbf{V}$ is the matrix appearing in the Fisher information matrix $\mathbf{J}(\gamma)$, $\gamma = \big(\boldsymbol{\mu}^T, e(\boldsymbol{\Sigma})^T\big)^T$, described by (14)*; for $q = 2$ the parameter* (17)

$$\lambda = p_1 p_2 \Big[(\mathbf{M}_1 - \mathbf{M}_2)^T \boldsymbol{\Sigma}^{-1} (\mathbf{M}_1 - \mathbf{M}_2) + e(\mathbf{W}_1 - \mathbf{W}_2)^T \mathbf{V} e(\mathbf{W}_1 - \mathbf{W}_2)\Big]\,. \tag{19}$$

If $k = 1$, then $V = 1/(2\Sigma_{11}^2)$, and (16) *holds with $(q-1)m = 2(q-1)$ and*

$$\lambda = \sum_{j=1}^{q} p_j \frac{\big(M_j - \overline{M}\big)^2}{\Sigma_{11}} + \sum_{j=1}^{q} p_j \frac{\big(W_j - \overline{W}\big)^2}{2\Sigma_{11}^2}\,. \tag{20}$$

Proof. Throughout the proofs assume that the Cartesian product is written not as rows, but as columns, i.e.,

$$\theta = \Big(\theta_1^T, \ldots, \theta_q^T\Big)^T, \quad \Theta = \Xi^q = \left\{\Big(\theta_1^{*T}, \ldots, \theta_q^{*T}\Big)^T;\ \theta_j^{*T} \in \Xi\,,\ j = 1, \ldots, q\right\}. \tag{21}$$

Since in (5) and (15) the functions $g_j(\theta) = \pi_j(\theta) - \pi_{j+1}(\theta)$, the matrix (6)

$$\boldsymbol{\partial}_{\mathbf{0}}(\theta) = \mathbf{U} \otimes I_m\,,$$

where the $(q-1) \times q$ matrix $\mathbf{U}$ has the elements $U_{ij} = 1$ if $i = j$, $U_{ij} = -1$ if $j = i+1$ and $U_{ij} = 0$ otherwise, and $k_0 = \operatorname{rank}\big(\boldsymbol{\partial}_{\mathbf{0}}(\theta)\big) = m(q-1)$. But the formula on MLE for one sample implies that the MLE under the null hypothesis is consistent, which together with Lemma 1 means that (10) and (11) hold with

$$h = \big(h1^T, \ldots, h_q^T\big)^T, \quad h_j = \Big(\mathbf{M}_j^T, e\big(\mathbf{W}_j\big)^T\Big)^T, \quad j = 1, \ldots, q. \tag{22}$$

As $\mathbf{J}(\theta) = I_q \otimes \mathbf{J}$, where $\mathbf{J}$ is the matrix (14), after some computation one obtains

$$\mathbf{F}_0 \mathbf{J}^{-1}(\theta) \mathbf{F}_0^T = \mathbf{L} \otimes \mathbf{J}^{-1} \,,$$

where the symmetric $(q-1) \times (q-1)$ matrix $\mathbf{L}$ has the elements

$$L_{ij} = \frac{1}{p_i} + \frac{1}{p_{i+1}} \text{ if } i = j, L_{ij} = -\frac{1}{p_j} \text{ if } j = i+1, \text{ and } L_{ij} = -\frac{1}{p_i} \text{ if } j = i-1$$

and $L_{ij} = 0$ otherwise. Hence

$$\left(\mathbf{F}_0 \mathbf{J}^{-1}(\theta) \mathbf{F}_0^T \right)^{-1} = \mathbf{L}^{-1} \otimes \mathbf{J}, \qquad \mathbf{L}^{-1} = \mathbf{C} \,,$$

where the matrix $\mathbf{C}$ has the elements

$$C_{ij} = \sum_{s=1}^{j} p_s \sum_{t=i+1}^{q} p_t \quad \text{if} \quad 1 \le j \le i \le q-1$$

and

$$C_{ij} = \sum_{t=j+1}^{q} p_t \sum_{s=1}^{i} p_s \quad \text{if} \quad 1 \le i \le j \le q-1.$$

We see that

$$\boldsymbol{\partial_0}(\theta)^T \left(\mathbf{F}_0 \mathbf{J}^{-1}(\theta) \mathbf{F}_0^T \right)^{-1} \boldsymbol{\partial_0}(\theta) = \big(\mathbf{U} \otimes I_m \big)^T \big(\mathbf{C} \otimes \mathbf{J} \big) \big(\mathbf{U} \otimes I_m \big) = \big(\mathbf{U}^T \mathbf{C} \mathbf{U} \big) \otimes \mathbf{J} \,, \tag{23}$$

$$\mathbf{U}^T \mathbf{C} \mathbf{U} = \operatorname{diag}(\mathbf{p}) - \mathbf{p}\mathbf{p}^T, \qquad \mathbf{p} = (p_1, \ldots, p_q)^T \tag{24}$$

and substituting (23), (24) and (22) into (11) and employing the formula (14) one obtains (17), the rest of the proof is obvious. □

Usually the one-dimensional normal distributions are described by the densities and the parameter set

$$f(x, \mu, \sigma) = \left(\sqrt{2\pi}\sigma \right)^{-1} \exp\big(-0.5(x-\mu)^2/\sigma^2 \big), \qquad \Xi = \big\{ (\mu, \sigma)' \in R^2;\, \sigma > 0 \big\} \,. \tag{25}$$

If the Pitman alternatives $\theta_u = \theta + \delta_u$, where $\theta = (\mu, \sigma, \ldots, \mu, \sigma)' \in \Xi^q$ and $\pi_j(\delta_u) = (M_j, W_j)'/\sqrt{N_u}$, then $\big(\sigma + W_j/\sqrt{N_u}\big)^2 = \sigma^2 + 2\sigma W_j/\sqrt{N_u} + o\Big(N_u^{-1/2}\Big)$ and the application of (7)–(10) and of (20) of Theorem 2 yields that

$$\lambda = \sum_{j=1}^{q} p_j \frac{\big(M_j - \overline{M} \big)^2}{\sigma^2} + 2 \sum_{j=1}^{q} p_j \frac{\big(W_j - \overline{W} \big)^2}{\sigma^2} \,. \tag{26}$$

The next theorem deals with testing the equality of means of the normal distributions without any assumptions on their variances, which is known as the Behrens-Fisher problem.

Theorem 3. *Suppose that the null hypothesis*

$$\Omega = \Omega_{BF} = \big\{(\theta_1, \ldots, \theta_q);\ \boldsymbol{\mu}_1 = \ldots = \boldsymbol{\mu}_q\big\} \tag{27}$$

and that $\theta = (\theta_1, \ldots, \theta_q) \in \Omega$, *i.e.,* θ_j *is the parameter of* $N_k(\boldsymbol{\mu}, \boldsymbol{\Sigma}_j)$ *distribution.*

(I) Let $k = 1$, *i.e., the normal distributions are defined on the real line. If the Pitman alternatives*

$$\theta + \delta_u\,, \quad \delta_u = \frac{1}{\sqrt{N_u}} \left(\begin{pmatrix} M_1 \\ W_1 \end{pmatrix}, \ldots, \begin{pmatrix} M_q \\ W_q \end{pmatrix} \right),$$

where M_j, W_j, $j = 1, \ldots, k$ *are real numbers, then for the LR statistic* (cf. (9), (8))

$$\mathcal{L}\Big[T_u \,\big|\, P_u^*\Big] \longrightarrow \chi^2_{(q-1)}(\lambda) \tag{28}$$

as $u \to \infty$, *and the noncentrality parameter*

$$\lambda = \sum_{j=1}^{q} p_j \frac{\big(M_j - \overline{M}\big)^2}{\Sigma_j}\,, \quad \overline{M} = \frac{1}{d} \sum_{j=1}^{q} p_j \frac{M_j}{\Sigma_j}\,, \quad d = \sum_{j=1}^{q} \frac{p_j}{\Sigma_j}\,. \tag{29}$$

If $q = 3$, *then*

$$\lambda = \frac{\frac{\Sigma_3}{p_3}(M_1 - M_2)^2 + \frac{\Sigma_2}{p_2}(M_1 - M_3)^2 + \frac{\Sigma_1}{p_1}(M_2 - M_3)^2}{\frac{\Sigma_1\Sigma_2}{p_1p_2} + \frac{\Sigma_1\Sigma_3}{p_1p_3} + \frac{\Sigma_2\Sigma_3}{p_2p_3}} \tag{30}$$

and if $q = 2$, *then*

$$\lambda = \frac{(M_1 - M_2)^2}{\frac{\Sigma_1}{p_1} + \frac{\Sigma_2}{p_2}}\,. \tag{31}$$

(II) Do not assume that $k = 1$. *Let* $q = 2$. *If the Pitman alternatives*

$$\theta_u = \theta + \delta_u\,, \quad \delta_u = \frac{1}{\sqrt{N_u}} \left(\begin{pmatrix} \mathbf{M}_1 \\ e(\mathbf{W}_1) \end{pmatrix}, \begin{pmatrix} \mathbf{M}_2 \\ e(\mathbf{W}_2) \end{pmatrix} \right), \tag{32}$$

where $\mathbf{M}_j \in R^k$ *and* $\mathbf{W}_j$ *is a symmetric* $k \times k$ *matrix,* $j = 1, 2$, *then for the LR statistic* (cf. (9), (8))

$$\mathcal{L}\Big[T_u \,\big|\, P_u^*\Big] \longrightarrow \chi^2_k(\lambda)\,, \tag{33}$$

where

$$\lambda = (\mathbf{M}_1 - \mathbf{M}_2)' \left(\frac{\boldsymbol{\Sigma}_1}{p_1} + \frac{\boldsymbol{\Sigma}_2}{p_2} \right)^{-1} (\mathbf{M}_1 - \mathbf{M}_2)\,. \tag{34}$$

Proof. Since the explicit formula for the MLE of the parameters from Ω_{BF} is not known, the condition of consistency of the MLE of the parameter from (27) under validity of the null hypothesis cannot be verified directly. However, if $A_{n_1,\ldots,n_q}$ denotes the set of realizations of the pooled sample $x_{(n_1,\ldots,n_q)}$ for which the sample covariance matrix S_j of the jth sample, $j = 1, \ldots, q$, is positive definite, then according to Corollary 1.4 on p. 53 of [7] there exists measurable MLE under the restriction Ω_{BF} defined on $A_{n_1,\dots,n_q}$ and any measurable version of this MLE is consistent; hence the condition on consistency of MLE of the unknown parameter from (27) is fulfilled.

(I) Use the notation (21) and (5) to describe (27). Then the matrix (6)

$$\boldsymbol{\partial}_{\mathbf{0}}(\theta) = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & \dots & 0 & 0 & 0 & 0 \\ \vdots & & & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 & -1 & 0 \end{pmatrix} \tag{35}$$

is of full rank $k_0 = q-1$, which together with Lemma 1 means that (10) and (11) hold with

$$h = \left(h1^T, \dots, h_q^T\right)^T, \quad h_j = (M_j, W_j)^T, \qquad j = 1, \dots, q. \tag{36}$$

As $\mathbf{J}(\theta) = I_q \otimes \mathbf{J}$, where $\mathbf{J}$ is the matrix (14), putting $\Sigma_i^* = \Sigma_i/p_i$ after some computation one obtains that the $(q-1)\times(q-1)$ matrix

$$\mathbf{F}_0 \mathbf{J}^{-1}(\theta)\mathbf{F}_0^T = \mathbf{L}\,,$$

where the matrix $\mathbf{L}$ has the elements $L_{ij} = -\Sigma_j^*$ if $j = i+1$, $L_{ij} = \Sigma_i^* + \Sigma_{i+1}^*$ if $j = i$, $L_{ij} = -\Sigma_i^*$ if $j = i-1$ and $L_{ij} = 0$ otherwise. Hence

$$\mathbf{L}^{-1} = \frac{1}{d}\mathbf{G}\,, \;\; d = \sum_{i=1}^{q} \frac{1}{\Sigma_i^*}\,, \;\; G_{ij} = \begin{cases} \sum\limits_{s=1}^{j} \frac{1}{\Sigma_s^*} \sum\limits_{t=i+1}^{q} \frac{1}{\Sigma_t^*}, & 1 \le j \le i \le q-1\,, \\ \sum\limits_{t=j+1}^{q} \frac{1}{\Sigma_t^*} \sum\limits_{s=1}^{i} \frac{1}{\Sigma_s^*}, & 1 \le i \le j \le q-1\,, \end{cases}$$

the matrix

$$\mathbf{S} = \boldsymbol{\partial}_{\mathbf{0}}(\theta)^T \mathbf{L}^{-1} \boldsymbol{\partial}_{\mathbf{0}}(\theta) \tag{37}$$

has the elements

$$S_{rt} = \begin{cases} 0 & \textit{if at least one of the integers } r, t \in \{1, \dots, 2q\} \textit{ is even,} \\ \frac{1}{\Sigma_i^*} - \frac{1}{d(\Sigma_i^*)^2}, & r = t = 2i-1\,,\; i = 1, \dots, q\,, \\ -\frac{1}{d\Sigma_i^*\Sigma_j^*}, & r = 2i-1\,, t = 2j-1\,, r \ne t\,. \end{cases} \tag{38}$$

Substituting (35)–(38) into (11) after some computation one obtains that (28) holds with (29).

(II) The proof is similar to the previous case. Obviously

$$\boldsymbol{\partial}_{\mathbf{0}}(\theta) = (I_k, \mathbf{0}, -I_k, \mathbf{0})\,, \quad \mathbf{F}_0\mathbf{J}^{-1}(\theta)\mathbf{F}_0^T = \frac{\boldsymbol{\Sigma}_1}{p_1} + \frac{\boldsymbol{\Sigma}_2}{p_2}\,,$$

which together with (32), (10) and (11) implies (I). □

One may conjecture that for the general dimension k the 1-dimensional formula (29) holds not only for $q = 2$ as stated in (34), but as the multiplication of matrices is not commutative, the steps of the proof of (29) cannot be simply extended to this general multidimensional setting when $q > 2$.

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